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ECCENTRIC AND SUPER ECCENTRIC SYMMETRIC n-SIGRAPHS

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Abstract

In this paper we introduced the new notions eccentric and super eccentric symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some switching equivalent characterizations.

1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [2]. We consider only finite, simple graphs free from self-loops.

Key Words : *Symmetric n-sigraph, Symmetric n-marked graph, Balance, Switching, Eccentric symmetric n-sigraph, Super eccentric symmetric n-sigraph, Complementation.*

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Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [10], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [6]).

Definition : Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [10].

Theorem 1.1 (**E. Sampathkumar et al. [10]**) : An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

In [10], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [4], [7-9], [12-22]).

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result (see [10]).

Theorem 1.2 (**E. Sampathkumar et al. [10]**) : Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Theorem 1.1.

In a graph G, the distance $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined by $r(G) = \min\{e(u) : u \in G\}$ and the diameter $d(G)$ of G is defined by $d(G) = \max\{e(u) : u \in G\}$. A graph for which $r(G) = d(G)$ is called a *self-centered graph* of radius $r(G)$.

Let $G = (V, E)$ be a simple undirected graph. The eccentricity $e(v)$ of a vertex in $V(G)$ is defined by $e(v) = \max_{u \in V} d(u, v)$, where $d(u, v)$ stands for the length of the shortest path in G between u and v. In case G is disconnected and u and v belong to different components, we set $d(u, v) = +\infty$.

Akiyama et al. [1] defined the eccentric graph $\mathcal{E}(G)$ of G as a graph on the same set of vertices as G obtained, by joining two vertices if and only if $d(u, v) = \min\{e(u), e(v)\}$.

Iqbalunnisa et al. [3] defined the super eccentric graph $\mathcal{SE}(G)$ of a graph G on the same set of vertices as G where the adjacency relation between vertices is defined by

$d(u,v) \geq \text{rad}(G)$ while G is connected and when G is disconnected, two vertices are adjacent in $\mathcal{SE}(G)$ if they belong to different components of G.

2. Eccentric n-Sigraph of an n-Sigraph

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of eccentric graphs to n-sigraphs as follows:

The *eccentric n-sigraph* $\mathcal{E}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{E}(G)$ and the n-tuple of any edge uv is $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called eccentric n-sigraph, if $S_n \cong \mathcal{E}(S'_n)$ for some n-sigraph S'_n. The following result restricts the class of eccentric graphs.

Theorem 2.1 : For any n-sigraph $S_n = (G, \sigma)$, its eccentric n-sigraph $\mathcal{E}(S_n)$ is i-balanced.

Proof : Since the n-tuple of any edge uv in $\mathcal{E}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1.1, $\mathcal{E}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated eccentric n-sigraph $\mathcal{E}(S_n)$ of S_n is defined as follows:

$$(\mathcal{E})^0(S_n) = S_n,\ (\mathcal{E})^k(S_n) = \mathcal{E}((\mathcal{E})^{k-1}(S_n)).$$

Corollary 2.2 : For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{E})^k(S_n)$ is i-balanced.

The following result characterize n-sigraphs which are eccentric n-sigraphs.

Theorem 2.3 : An n-sigraph $S_n = (G, \sigma)$ is an eccentric n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is an eccentric graph.

Proof : Suppose that S_n is i-balanced and G is a $\mathcal{E}(G)$. Then there exists a graph H such that $\mathcal{E}(H) \cong G$. Since S_n is i-balanced, by Theorem 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{E}(S'_n) \cong S_n$. Hence S_n is an eccentric n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an eccentric n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{E}(S'_n) \cong S_n$. Hence G is the $\mathcal{E}(G)$ of H and by Theorem 2.1, S_n is i-balanced. □

Let $S_i = \{v \in V(G) | e(v) = i\}$, $i = 1, 2, \cdots$. In [?], the authors completely characterize those graphs whose eccentric graph is isomorphic to its complement.

Theorem 2.4 : $\mathcal{E}(G) \cong \overline{G}$ if and only if $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.

In view of the above result, we have the following result that characterizes the family of n-sigraphs satisfies $\mathcal{E}(S_n) \sim \overline{S_n}$.

Theorem 2.5 : For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{E}(S_n) \sim \overline{S_n}$ if, and only if, G is a graph with $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.

Proof : Suppose that $\mathcal{E}(S_n) \sim \overline{S_n}$. Then clearly, $\mathcal{E}(G) \cong \overline{G}$. Hence by Theorem 2.4, G is a graph with $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour.

Conversely, suppose that S_n is an n-sigraph whose underlying graph G is a graph $S_i = \phi$, $i = 1, 4, 5, 6, \cdots$ and no two vertices in S_3 have a common nieghbour. Then by Theorem 2.4, $\mathcal{E}(G) \cong \overline{G}$. Since for any n-sigraph S_n, both $\mathcal{E}(S_n)$ and $\bar{(}S_n)$ are i-balanced, the result follows by Theorem 1.2. □

The following result characterizes the n-sigraphs which are cycle isomorphic to eccentric n-sigraphs. In case of graphs the following result is due to Akiyama et al. [1] :

Theorem 2.6 : If $r(G) = 1$, then $\mathcal{E}(G) \cong G$ if and only if $< V - S_1 >_G$ is self-complementary, where S_1 denotes the set of vertices in G of eccentricity 1.

Theorem 2.7 : An n-sigraph $S_n = (G, \sigma)$ with $r(G) = 1$, $S_n \sim \mathcal{E}(S_n)$ if, and only if, S_n is i-balanced and $< V - S_1 >_G$ is self-complementary, where S_1 denotes the set of vertices in G of eccentricity 1.

Proof : Suppose $\mathcal{E}(S_n) \sim S_n$. This implies, $\mathcal{E}(G) \cong G$ and hence by Theorem 2.6, we see that the graph G satisfies the conditions in Theorem 2.6. Now, if S_n is any n-sigraph with $< V - S_1 >_G$ is self-complementary, where S_1 denotes the set of vertices in G of eccentricity 1, Theorem 2.1 implies that $\mathcal{E}(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its eccentric n-sigraph $\mathcal{E}(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is i-balanced n-sigraph with $< V - S_1 >_G$ is self-complementary, where S_1 denotes the set of vertices in G of eccentricity 1. Then, since $\mathcal{E}(S_n)$ is i-balanced as per Theorem 2.1 and since $\mathcal{E}(G) \cong G$ by Theorem 2.6, the result follows from Theorem 1.2 again. □

3. Super Eccentric n-Sigraph of an n-Sigraph

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of super eccentric graphs to n-sigraphs as follows:

The *super eccentric n-sigraph* $\mathcal{SE}(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $\mathcal{SE}(G)$ and the n-tuple of any edge uv is $\mathcal{SE}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called super eccentric n-sigraph, if $S_n \cong \mathcal{SE}(S'_n)$ for some n-sigraph S'_n. The following result restricts the class of super eccentric graphs.

Theorem 3.1 : For any n-sigraph $S_n = (G, \sigma)$, its super eccentric n-sigraph $\mathcal{SE}(S_n)$ is i-balanced.

Proof : Since the n-tuple of any edge uv in $\mathcal{SE}(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1.1, $\mathcal{SE}(S_n)$ is i-balanced. □

For any positive integer k, the k^{th} iterated super eccentric n-sigraph $\mathcal{SE}(S_n)$ of S_n is defined as follows:

$$(\mathcal{SE})^0(S_n) = S_n,\ (\mathcal{SE})^k(S_n) = \mathcal{SE}((\mathcal{SE})^{k-1}(S_n)).$$

Corollary 3.2 : For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(\mathcal{SE})^k(S_n)$ is i-balanced.

The following result characterize n-sigraphs which are super eccentric n-sigraphs.

Theorem 3.3 : An n-sigraph $S_n = (G, \sigma)$ is a super eccentric n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a super eccentric graph.

Proof : Suppose that S_n is i-balanced and G is a $\mathcal{SE}(G)$. Then there exists a graph H such that $\mathcal{SE}(H) \cong G$. Since S_n is i-balanced, by Theorem 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{SE}(S'_n) \cong S_n$. Hence S_n is a super eccentric n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a super eccentric n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{SE}(S'_n) \cong S_n$. Hence G is the $\mathcal{SE}(G)$ of H and by Theorem 2.1, S_n is i-balanced. □

In [5], the author characterize those graphs whose super eccentric graph is isomorphic to its complement.

Theorem 3.4 : For any graph G, $\mathcal{SE}(G) \cong \overline{G}$ if and only if $r(G) = 2$ or G is disconnected with each component complete.

In view of the above result, we have the following result that characterizes the family of n-sigraphs satisfies $\mathcal{SE}(S_n) \sim \overline{S_n}$.

Theorem 3.5 : For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{SE}(S_n) \sim \overline{S_n}$ if, and only if, G is a graph with $r(G) = 2$ or G is disconnected with each component complete.

Proof : Suppose that $\mathcal{SE}(S_n) \sim \overline{S_n}$. Then clearly, $\mathcal{SE}(G) \cong \overline{G}$. Hence by Theorem 3.4, G is a graph with $r(G) = 2$ or G is disconnected with each component complete.

Conversely, suppose that S_n is an n-sigraph whose underlying graph G is a graph with $r(G) = 2$ or G is disconnected with each component complete. Then by Theorem 3.4, $\mathcal{SE}(G) \cong \overline{G}$. Since for any n-sigraph S_n, both $\mathcal{SE}(S_n)$ and $\overline{S}_n$ are i-balanced, the result follows by Theorem 1.2. □

4. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $\mathcal{DCP}(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $\mathcal{DCP}(S_n)$ is i-balanced, where for any $m \in H_n$. For an n-sigraph $S_n = (G, \sigma)$, the $\mathcal{E}(S_n)$ and $\mathcal{SE}(S_n)$ are i-balanced. We now examine, the conditions under which m-complement of $\mathcal{E}(S_n)$ and $\mathcal{SE}(S_n)$ are i-balanced, where for any $m \in H_n$.

Theorem 4.1 : Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $\mathcal{E}(G)$ ($\mathcal{SE}(G)$) is bipartite then $(\mathcal{E}(S_n))^m$ $((\mathcal{SE}(S_n))^m)$ is i-balanced.

Proof : Since, by Theorem 2.1 (Theorem 3.1), $\mathcal{E}(S_n)$ ($\mathcal{SE}(S_n)$) is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ ($\mathcal{SE}(S_n)$) whose k^{th} co-ordinate are $-$ is even. Also, since $\mathcal{E}(G)$ ($\mathcal{SE}(G)$) is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $\mathcal{E}(S_n)$ $(\mathcal{SE}(S_n))$ whose k^{th} co-ordinate are + is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(\mathcal{E}(S_n))^t$ $((\mathcal{SE}(S_n))^t)$ is i-balanced. □

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